Relative entropy of entanglement of rotationally invariant states

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We calculate the relative entropy of entanglement for rotationally invariant states of spin-$\frac{1}{2}$ and arbitrary spin-j particles or of spin-1 particle and spin-j particle with integer j. A lower bound of relative entropy of entanglement and an upper bound of distillable entanglement are presented for rotationally invariant states of spin-1 particle and spin-j particle with half-integer j.

I.   INTRODUCTION

Quantum entanglement has played a significant role in the field of quantum information and quantum computation [1]. This attracts an increasing interest in the study of quantification of entanglement for any quantum state. Although a lot of entanglement measures have been proposed for a generic mixed state, only partial solutions such as entanglement of formation (EoF) for two qubits [2, 3], are known to give the closed forms for generic bipartite states. In particular, symmetric states have elegant forms to quantify their entanglement. For example, E.M. Rains [4] and Y.X. Chen *et. al* [5] calculate the relative entropy of entanglement (REE) for maximally correlated state by use of different ways. In addition, P. Rungta *et. al.* obtained concurrence for isotropic states in [6] and K. Chen *et. al* presented tangle and concurrence for Werner states in [7].

As invariant states under $SO(3)$ group have a relatively simple structure and therefore have been investigated extensively in the literature [8, 9, 10, 11, 12, 13, 14]. This family of symmetric states under $SO(3)$ group is called rotationally invariant (RI) states. K.K. Manne and C.M. Caves [15] derived an analytic expression for the EoF, I-concurrence, I-tangle and convex-roof-extended negativity of RI states of a spin-j particle and spin-

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$\frac{1}{2}$ particle by using K.G.H. Vollbrecht and R.F. Werner's method [16]. It is known that the REE is one of the fundamental entanglement measures, as relative entropy is one of the most important functions in quantum information theory. Therefore, in this paper we apply K.G.H. Vollbrecht and R.F. Werner's technique to derive the relative entropy of entanglement for RI states of a spin-$\frac{1}{2}$ particle and arbitrary spin-j particle or of a spin-1 particle and a spin-j particle with integer j.

This paper is organized as follows. In section II at first we review the definition and some properties of REE and the representations of RI states for two particles. Then we show the simplified expression of REE on RI states. In section III we calculate the REE explicitly for RI state of spin-$\frac{1}{2}$ and spin-j particles. The REE is compared for different spin-j particle. In section IV at first we obtain the relative entropy of entanglement for RI state in the system of two spin-1 particels. Subsequently, we introduce a way to obtain the separable state which minimizes the REE for RI state of the case $j_1 = 1$, $j_2 = 2$. Furthermore, this result can be extended to the case of spin-1 particle and spin-j particle with integer j. Thus we obtain the REE for this family of RI states. Finally, a lower bound of REE and an upper bound of distillable entanglement are presented for RI states of spin-1 particle and spin-j particle with half-integer j. In section V a few conclusions are drawn.

II. PRELIMINARIES

Throughout this paper we refer to $\mathcal{S}$, $\mathcal{D}$ and $\mathcal{P}$ as the set of all states, separable states and positive partial transposition (PPT) states, respectively. Relative entropy [17, 18] of entanglement is defined as

$$E_r(\rho) = \min_{\sigma \in \mathcal{D}} S(\rho\|\sigma) = \min_{\sigma \in \mathcal{D}} \mathrm{tr}(\rho \ln \rho - \rho \ln \sigma). \tag{1}$$

There are some properties of REE as follows [4, 18]:

(1) $E_r(\rho) \geq 0$ with the equality saturated iff ρ is a separable state.

(2) Local unitary operations leave $E_r(\rho)$ invariant.

(3) $E_r(\rho)$ cannot increase under LOCC.

(4) For a pure state ρ we have $E_r(\rho) = S(\rho_A) = -\mathrm{tr}(\rho_A \ln \rho_A)$, where $S(\rho_A)$ is the entropy of entanglement of $\rho_A = \mathrm{tr}_B(\rho)$, tr_B is a map of operators known as the partial trace over system B.

(5) If σ^* minimizes $S(\rho\|\sigma^*)$ over $\sigma \in \mathcal{D}$ then σ^* is also a minimum for any state of the form $\rho_x = (1-x)\rho + x\sigma^*$.

(6) $E_r(x_1\rho_1 + x_2\rho_2) \leq x_1 E_r(\rho_1) + x_2 E_r(\rho_2)$, where $x_1 + x_2 = 1$ and x_1, x_2 are non-negative and real.

(7) $E_r(\rho) \leq E_F(\rho)$, where $E_F(\rho)$ is EoF.

(8) $E_D(\rho) \leq E_\Gamma(\rho) \leq E_r(\rho)$, where $E_D(\rho)$ is distillable entanglement and $E_\Gamma(\rho) = \min_{\sigma \in \mathcal{P}} S(\rho\|\sigma)$, Γ denotes the partial transpose operator.

Now we recall the representations of RI states for two particles with spins j_1 and j_2 and corresponding angular momentum operators $\hat{j}^1$ and $\hat{j}^2$ [8, 9]. Throughout this paper we will assume that $j_2 \geq j_1$. The tensor product $\mathbb{C}^{N_1} \otimes \mathbb{C}^{N_2}$ is the Hilbert space of a system which is composed of spin-j_1 particle and spin-j_2 particle. The Hilbert space $\mathbb{C}^{N_1}$ of the first space is spanned by the common eigenvectors $|j_1, m_1\rangle$ of the square of $\hat{j}^1$ and of $\hat{j}_z^1$, where $N_1 = 2j_1 + 1$ and $m_1 = -j_1, \cdots, +j_1$. The Hilbert space $\mathbb{C}^{N_2}$ of the second space is spanned by the eigenvectores $|j_2, m_2\rangle$, where $N_2 = 2j_2 + 1$ and $m_2 = -j_2, \cdots, +j_2$, correspondingly. H.P. Breuer [10, 11] considered the representation of RI states which employs the projection operators $P_J = \displaystyle\sum_{M=-J}^{J} |JM\rangle\langle JM|$. Notice that here $|JM\rangle$ is the common eigenvector of the square of the total angular momentum operator and of its z-component. RI states using the representation can be written as

$$\rho = \frac{1}{\sqrt{N_1 N_2}} \sum_{J=j_2-j_1}^{j_1+j_2} \frac{\alpha_J}{\sqrt{2J+1}} P_J, \tag{2}$$

where the α_J are real parameters and $\sqrt{N_1 N_2}$ and $\sqrt{2J+1}$ are introduced as convenient normalization factors. In order for ρ to represent a density matrix the α_J must be positive and normalized appropriately:

$$\alpha_J \geq 0, \qquad \mathrm{tr}\rho = \sum_J \sqrt{\frac{2J+1}{N_1 N_2}} \alpha_J = 1. \tag{3}$$

We denote the set of all vectors $\vec{\alpha}$ whose components α_J satisfy the relations (3) by S^α. It is obvious that S^α is isomorphic to the set of RI states and of course a convex set. It is remarkable that the representation (2) is the spectral decomposition of ρ.

In the following we present the simplified expression of REE on RI states. What makes the calculation of the REE easy for RI states is the existence of a "twirl" operation [19], a projection operator $\mathbf{P}$ that maps an arbitrary state ρ to a RI state $\mathbf{P}(\rho)$ and that preserves separability, i.e., that maps every separable state to a RI separable state. Since for a RI state ρ we have

$$S(\rho\|\sigma) \geq S(\mathbf{P}(\rho)\|\mathbf{P}(\sigma)),$$

this guarantees that the minimum REE for a RI state is attained on another RI separable state [16]. Hence, suppose

$$\rho^* = \frac{1}{\sqrt{N_1 N_2}} \sum_{J=j_2-j_1}^{j_1+j_2} \frac{\alpha_J{}^*}{\sqrt{2J+1}} P_J$$

be a RI separable state, one can show

$$E_r(\rho) = \min \sum_{J=j_2-j_1}^{j_1+j_2} \sqrt{\frac{2J+1}{N_1 N_2}} \alpha_J (\ln \alpha_J - \ln \alpha_J^*), \tag{4}$$

where we utilize the fact that equation (2) is the spectral decomposition of RI state ρ.

III.   REE FOR $2 \otimes N$ SYSTEM

For a bipartite system consisting of a spin-$\frac{1}{2}$ particle and a spin-j particle, the RI state ρ can be written as a function of a single parameter p:

$$\rho = \frac{p}{2j} \sum_{m=-j+\frac{1}{2}}^{j-\frac{1}{2}} |j-\tfrac{1}{2}, m\rangle\langle j-\tfrac{1}{2}, m| + \frac{1-p}{2j+2} \sum_{m=-j-\frac{1}{2}}^{j+\frac{1}{2}} |j+\tfrac{1}{2}, m\rangle\langle j+\tfrac{1}{2}, m|. \tag{5}$$

This equation is the spectral decomposition of ρ with the eigenvalues p and $1-p$. From [8] we know that ρ is separable iff $p \leq \frac{2j}{2j+1}$. It is clear that $E_r(\rho) = 0$ for the states with $p \leq \frac{2j}{2j+1}$. For this family of RI states the set of separable states is just interval and the definition of REE requires a minimization over this interval. Thus for RI states of spin-$\frac{1}{2}$

and spin-j particles the minimizing separable state is the boundary state with $p = \frac{2j}{2j+1}$. It follows from equation (4) that

$$E_r(\rho) = \begin{cases} 0, & p \le \frac{2j}{2j+1}, \\ p\ln(\frac{2j+1}{2j}p) + (1-p)\ln[(2j+1)(1-p)], & p > \frac{2j}{2j+1}, \end{cases} \tag{6}$$

for a bipartite system of a spin-$\frac{1}{2}$ particle and a spin-j particle. We plot the $E_r(\rho)$ for $j = \frac{1}{2}$, 1, $\frac{3}{2}$ in Figure 1 which shows us that the RI states become less entangled as j increase. K.K. Manne and C.M. Caves obtained the same conclusion according to EoF in [15].
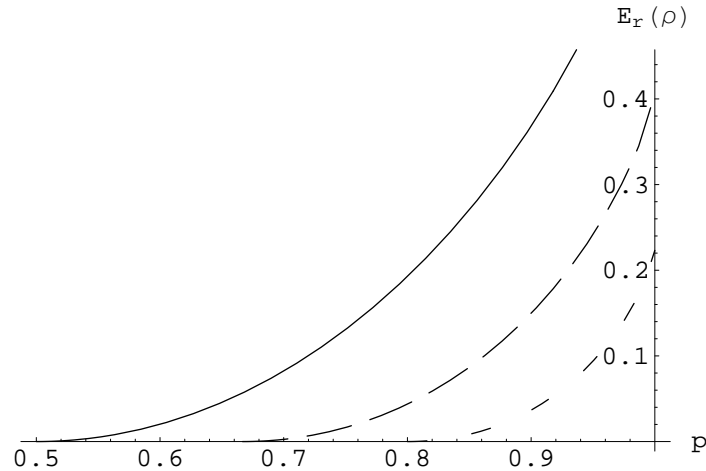

FIG. 1: The entanglement of relative entropy for $j = \frac{1}{2}$ (solid), $j = 1$ (long-dashed) and $j = \frac{3}{2}$ (short-dashed).

Interestingly, we find that our result for two spin $\frac{1}{2}$ particles coincides with the case of two spin $\frac{1}{2}$ particles (i.e. Werner states) with the asymptotic value obtained by K. Audenaert et $al.$ in [21].

IV.   REE FOR $3 \otimes N$ SYSTEM

In the section we discuss the REE of RI states of $j_1 = 1(N_1 = 3)$ particle and arbitrary j_2 particle, that is, of $3 \otimes N_2$ system. Set $j = j_2$ and $N = N_2 = 2j_2 + 1$ for convenience. Since J takes on the values $J = j - 1$, j and $j + 1$, $\vec{\alpha}$ is a three-vector $\vec{\alpha} = (\alpha_{j-1} \ \ \alpha_j \ \ \alpha_{j+1})^T$.

From equation (3) we know the set of RI states is given by the relations: $\alpha_{j-1}, \alpha_j, \alpha_{j+1} \geq 0$ and

$$\sqrt{\frac{N-2}{3N}}\alpha_{j-1} + \sqrt{\frac{1}{3}}\alpha_j + \sqrt{\frac{N+2}{3N}}\alpha_{j+1} = 1. \tag{7}$$

We infer from the equation (3) that S^α is a 2-simplex, i.e. a triangle with vertices as follows:

$$A = \begin{pmatrix} 0 \\ 0 \\ \sqrt{\frac{3N}{N+2}} \end{pmatrix}, \quad B = \begin{pmatrix} \sqrt{\frac{3N}{N-2}} \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ \sqrt{3} \\ 0 \end{pmatrix}. \tag{8}$$

In [10, 11] H.P. Breuer introduced the time reversal transformation ϑ which is unitarily equivalent to the transposition T. Therefore the Peres-Horodecki criterion can be expressed by $\vartheta_2\rho = (I \otimes \vartheta)\rho \geq 0$. It is worth mentioning that ϑ_2 is taken to be of the form $\vartheta_2(A \otimes B) = A \otimes \vartheta B = A \otimes VB^T V^\dagger$, where V is a unitary matrix which represents a rotation of the coordinate system about the y-axis by the angle π. It follows from [11] that $\vartheta_2 S^\alpha$ is also a 2-simplex with vertices

$$A' = \begin{pmatrix} \sqrt{\frac{3(N-2)}{N}} \\ \frac{2\sqrt{3}}{N+1} \\ \frac{2}{N+1}\sqrt{\frac{3}{N(N+2)}} \end{pmatrix}, \quad B' = \begin{pmatrix} \frac{2}{N-1}\sqrt{\frac{3}{N(N-2)}} \\ -\frac{2\sqrt{3}}{N-1} \\ \sqrt{\frac{3(N+2)}{N}} \end{pmatrix}, \quad C' = \begin{pmatrix} -\frac{2}{N-1}\sqrt{\frac{3(N-2)}{N}} \\ \sqrt{3}\frac{N^2-5}{N^2-1} \\ \frac{2}{N+1}\sqrt{\frac{3(N+2)}{N}} \end{pmatrix}.$$

which are the images of A, B and C under the action of ϑ_2, respectively. Relation (7) implies that we can represent the RI states using points in α-space by two coordinates (α_{j-1}, α_j). Consequently, $S^\alpha_{PPT}(= S^\alpha \cap \vartheta_2 S^\alpha)$ is a polygon with four vertices A, A', D and E, where A, A' are given by the above equations and

$$D = \begin{pmatrix} \frac{N-1}{2}\sqrt{\frac{3}{N(N-2)}} \\ 0 \\ \frac{N+1}{2}\sqrt{\frac{3}{N(N+2)}} \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ \sqrt{3}\frac{N-1}{N+1} \\ \frac{2}{N+1}\sqrt{\frac{3N}{N+2}} \end{pmatrix}.$$

H.P. Breuer [11] proved that PPT is necessary and sufficient for separability of all $3 \otimes N$ systems with odd N. So the polygon $ADA'E$ represents the set of separable states for a bipartite system of spin-1 particle and spin-j particle with integer j. Thus the separable state that minimizes any RI state of spin-1 particle and spin-j particle with integer j can be obtained. In the following we will discuss the REE for RI states of $3 \otimes N$ systems. Throughout this section we refer to the state corresponding to the point A as ρ_A (A stands for any letter).

A.   The case $3 \otimes 3$

The state space for $3 \otimes 3$ RI states is split naturally into four regions: the separable rectangle $ADA'E$ and the three triangles $A'CE$, $A'BD$ and $A'BC$ in Figure 2.
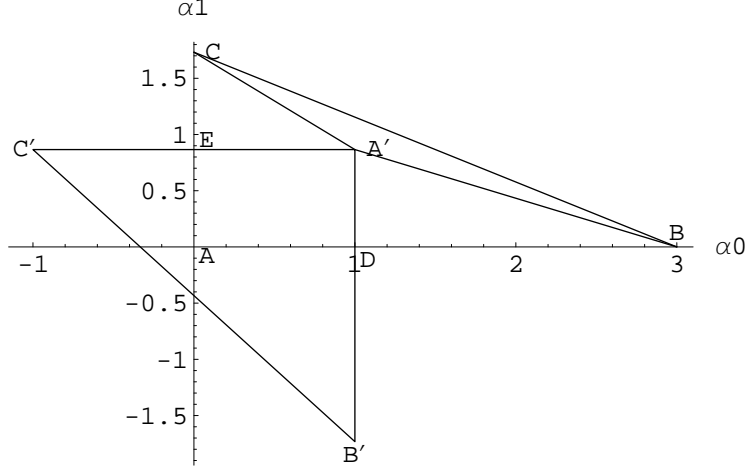

FIG. 2: $3 \otimes 3$ rotationally invariant states

It is obvious that $E_r(\rho) = 0$ for any state in the rectangle $ADA'E$. It is easy to see that the state ρ_C and the states on the whole line EA' are depicted by the vector $\vec{\alpha} = (0, \sqrt{3}, 0)^T$ and $\vec{\alpha} = (x, \frac{\sqrt{3}}{2}, \frac{3}{\sqrt{5}}(\frac{1}{2} - \frac{x}{3}))^T$, respectively. According to equation (4), one can easily show any state on the whole line connecting the points A' and E, is a separable state which minimizes the REE for ρ_C. As a result of the property (5) of REE, we can find the minimizing separable state for any state in the whole triangle $A'CE$. One just has to connect the point of given states with the point C to draw a straight line. The intersection with the line EA' is a minimizer for ρ_C and all states on the connecting line. Similarly, we can obtain the minimizing separable states for ρ_B and any state in the whole triangle $A'BD$. In addition, all the states in the whole triangle $A'BC$ have the same minimizer $\rho_{A'}$. In fact, one just has to show that the minimizer for any state on the line BC is the separable state $\rho_{A'}$ because of the property (5) of relative entropy of entanglement. In order to simplify the calculation, we write any $3 \otimes 3$ rotationally invariant state which is depicted by the vector $\vec{\alpha} = (3\alpha_0, \sqrt{3}\alpha_1, \frac{3}{\sqrt{5}}(1 - \alpha_0 - \alpha_1))^T$ as ρ, where $\alpha_0 \in [0, 1]$, $\alpha_1 \in [0, 1]$. Consequently, for RI state ρ in the system of two spin-1 particles we have

- If ρ is in the rectangle $ADA'E$, then $E_r(\rho) = 0$.

- If ρ is in the triangle $A'CE$, then $E_r(\rho) = (1 - \alpha_1) \ln[2(1 - \alpha_1)] + \alpha_1 \ln(2\alpha_1)$.

- If ρ is in the triangle $A'BC$, then

$$E_r(\rho) = \alpha_0 \ln(3\alpha_0) + \alpha_1 \ln(2\alpha_1) + (1 - \alpha_0 - \alpha_1) \ln[6(1 - \alpha_0 - \alpha_1)].$$

- If ρ is in the triangle $A'BD$, then $E_r(\rho) = \alpha_0 \ln(3\alpha_0) + (1 - \alpha_0) \ln[\frac{3}{2}(1 - \alpha_0)]$.

B.   The case $3 \otimes N$ with odd N

The state space for $3 \otimes 5$ RI states is split into four regions by the below discussions for REE: the separable polygon $ADA'E$, the entangled triangle $A'DH$, the entangled polygon $A'HBF$ and the entangled polygon $A'FCE$ in figure 3. The coordinates of the point F, G and H are $(\frac{\sqrt{5}}{2}, \frac{\sqrt{3}}{2})$, $(\frac{16\sqrt{5}}{25}, 0)$ and $(\frac{24\sqrt{5}}{25}, 0)$, respectively. Obviously, $E_r(\rho) = 0$ for any state in the polygon $ADA'E$.
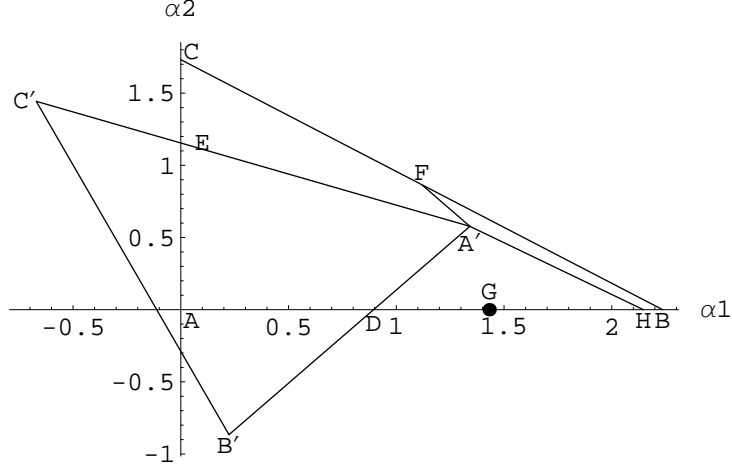

FIG. 3: $3 \otimes 5$ rotationally invariant states

One can write the states on the lines BC, $A'E$ and $A'D$ using the parameter vectors

$$\vec{\alpha} = \begin{pmatrix} a \\ -\sqrt{\frac{3}{5}} \cdot a + \sqrt{3} \\ 0 \end{pmatrix}, \ \vec{\alpha}_x = \begin{pmatrix} x \\ -\frac{\sqrt{15}}{9}x + \frac{2}{\sqrt{3}} \\ \sqrt{\frac{15}{7}}(\frac{1}{3} - \frac{4\sqrt{5}}{45}x) \end{pmatrix}, \ \vec{\alpha}_y = \begin{pmatrix} y \\ \frac{\sqrt{15}}{3}y - \frac{2}{\sqrt{3}} \\ \sqrt{\frac{15}{7}}(\frac{5}{3} - \frac{8\sqrt{5}}{15}y) \end{pmatrix},$$

respectively. We denote the corresponding state to any point on the whole line BC as ρ_a with $a \in [0, \sqrt{5}]$. Then from equation (4) we can obtain $E_r(\rho_a) = \min\{f(x), g(y)\}$, where

$$f(x) = \tfrac{a}{\sqrt{5}}(\ln a - \ln x) + (-\tfrac{a}{\sqrt{5}} + 1)[\ln(-\sqrt{\tfrac{3}{5}} \cdot a + \sqrt{3}) - \ln(-\tfrac{\sqrt{15}}{9}x + \tfrac{2}{\sqrt{3}})],$$
$$g(y) = \tfrac{a}{\sqrt{5}}(\ln a - \ln y) + (-\tfrac{a}{\sqrt{5}} + 1)[\ln(-\sqrt{\tfrac{3}{5}} \cdot a + \sqrt{3}) - \ln(\tfrac{\sqrt{15}}{3}y - \tfrac{2}{\sqrt{3}})],$$

with $x \in [0, \tfrac{3}{\sqrt{5}}]$, $y \in [\tfrac{2}{\sqrt{5}}, \tfrac{3}{\sqrt{5}}]$. Here $f(x)$ is a continuous function of x. Hence there must exist minimum for function $f(x)$ in the closed interval $[0, \tfrac{3}{\sqrt{5}}]$. Analogous to $f(x)$, there exists minimum for function $g(y)$ in the closed interval $[\tfrac{2}{\sqrt{5}}, \tfrac{3}{\sqrt{5}}]$. By a tedious calculation, we can obtain

$$E_r(\rho_a) = \begin{cases} f(\tfrac{6a}{5}), & a \in [0, \tfrac{\sqrt{5}}{2}], \\ f(\tfrac{3}{\sqrt{5}}), & a \in [\tfrac{\sqrt{5}}{2}, \sqrt{5}]. \end{cases}$$

Every state on the line BD can be represented by the parameter vector $\vec{\alpha} = (b,\, 0,\, \sqrt{\tfrac{15}{7}}(1 - \tfrac{b}{\sqrt{5}}))^T$. In the sequel, we label the corresponding state to each point on the whole line BC as ρ_b with $b \in [\tfrac{2}{\sqrt{5}}, \sqrt{5}]$. Then $E_r(\rho_b) = \min\{f(x), g(y)\}$, here

$$f(x) = \tfrac{b}{\sqrt{5}}(\ln b - \ln x) + (-\tfrac{b}{\sqrt{5}} + 1)\{\ln[\sqrt{\tfrac{15}{7}}(-\tfrac{b}{\sqrt{5}} + 1)] - \ln[\sqrt{\tfrac{15}{7}}(\tfrac{1}{3} - \tfrac{4\sqrt{5}}{45}x)]\},$$
$$g(y) = \tfrac{b}{\sqrt{5}}(\ln b - \ln y) + (-\tfrac{b}{\sqrt{5}} + 1)\{\ln[\sqrt{\tfrac{15}{7}}(-\tfrac{b}{\sqrt{5}} + 1)] - \ln[\sqrt{\tfrac{15}{7}}(\tfrac{5}{3} - \tfrac{8\sqrt{5}}{15}y)]\},$$

with $x \in [0, \tfrac{3}{\sqrt{5}}]$, $y \in [\tfrac{2}{\sqrt{5}}, \tfrac{3}{\sqrt{5}}]$. By a similar derivation, we can obtain

$$E_r(\rho_b) = \begin{cases} g(\tfrac{5b}{8}), & b \in [\tfrac{16\sqrt{5}}{25}, \tfrac{24\sqrt{5}}{25}], \\ g(\tfrac{3}{\sqrt{5}}), & b \in [\tfrac{24\sqrt{5}}{25}, \sqrt{5}]. \end{cases}$$

To summarize, we present that the separable states corresponding to the point $P = (\tfrac{6a}{5}, -\tfrac{2a}{\sqrt{15}} + \tfrac{2}{\sqrt{3}})$ and $Q = (\tfrac{5b}{8}, \tfrac{5\sqrt{15}}{24}b - \tfrac{2}{\sqrt{3}})$ minimize the states ρ_a with $a \in [0, \tfrac{\sqrt{5}}{2}]$ and ρ_b with $b \in [\tfrac{16\sqrt{5}}{25}, \tfrac{24\sqrt{5}}{25}]$, respectively. Accordingly, it is a remarkable fact that the state $\rho_{A'}$ is the minimizing separable state for the states ρ_a with $a \in [\tfrac{\sqrt{5}}{2}, \sqrt{5}]$ and ρ_b with $b \in [\tfrac{24\sqrt{5}}{25}, \sqrt{5}]$. We should emphasize that the states on the whole line DG have the same minimizing state ρ_D. The corresponding nearest separable state on the line $A'D$ approaches ρ_D as the state on the line HG approaches ρ_G. Hence, one may take the triangle $A'DH$ as the polygon $A'DGH$. According to the property (5) of REE, we can obtain the REE for any $3 \otimes 5$ RI state (see the below paragraph).

In much the same way as the above derivation, in terms of relative entropy of entanglement, the state space for $3 \otimes N$ with odd N RI states is split into four re-

gions: the separable polygon $ADA'E$, the entangled triangle $A'DH$, the entangled polygon $A'HBF$ and the entangled polygon $A'FCE$. The coordinates of F, G and H are $(\frac{N-3}{N-1}\sqrt{\frac{3N}{N-2}}, \frac{2}{N-1}\sqrt{3})$, $(\frac{(N-1)^2(N+3)}{2(N^2-5)}\sqrt{\frac{3}{N(N-2)}}, 0)$ and $(\frac{(N+3)(N-1)}{N^2-5}\sqrt{\frac{3(N-2)}{N}}, 0)$, respectively. It is easy to see that $E_r(\rho) = 0$ for any state in the polygon $ADA'E$. In addition, $\rho_{A'}$ is the minimizing state for any state of the polygon $A'HBF$. By a similar discussion with the case $3 \otimes 5$, we can obtain that the coordinates of P and Q are $(\frac{(N-2)(N-1)}{N(N-3)}a, \frac{N-1}{N+1}(-\sqrt{\frac{N-2}{N}}a + \sqrt{3}))$ and $(\frac{N^2-5}{(N+3)(N-1)}b, \frac{4(N^2-5)\sqrt{N(N-2)}}{(N^2-1)(N^2-9)}b - \frac{2\sqrt{3}(N-1)}{(N+1)(N-3)})$, respectively. All the states on the line connecting the point $(a, -\sqrt{\frac{N-2}{N}} \cdot a + \sqrt{3})$ with the point P have the same minimizing separable state ρ_P in the polygon $A'CEF$. Similarly, all the states on the line connecting the point $(b, 0)$ with the point Q have the same minimizing separable state ρ_Q in the triangle $A'DH$. It is worth mentioning that the separable state ρ_D minimizes all the states on the whole line DG. To simplify the calculation, suppose that an RI state of spin-1 particle and spin-j particle with integer j is depicted by the vector $\vec{\alpha} = (\sqrt{\frac{3N}{N-2}}\alpha_{j-1}, \sqrt{3}\alpha_j, \sqrt{\frac{3N}{N+2}}(1 - \alpha_{j-1} - \alpha_j))^T$, where $\alpha_{j-1} \in [0, 1]$, $\alpha_j \in [0, 1]$. Consequently, for RI state ρ in the system of a spin-1 particle and a spin-j particle with integer $j > 1$, we have

- If ρ is in the rectangle $ADA'E$, then $E_r(\rho) = 0$.

- If ρ is in the triangle $A'FCE$, then

$$E_r(\rho) = \alpha_{j-1} \ln \frac{N(N-3)\alpha_{j-1}}{(N-1)(N-2)a} + \alpha_j \ln \frac{(N+1)\alpha_j}{(N-1)(1-a)}$$

$$+(1 - \alpha_{j-1} - \alpha_j) \ln \frac{N(N+1)(N-3)(1-\alpha_{j-1}-\alpha_j)}{2[N(N-3)-(N-1)^2 a]},$$

here

$$a = \frac{-t_1 - \sqrt{t_1^2 - 4N(N-1)^2(N-3)\alpha_{j-1}}}{2(N-1)^2},$$

$$t_1 = (N+1)\alpha_j + N(N-3)\alpha_{j-1} - (N-1)^2.$$

- If ρ is in the triangle $A'HBF$,

$$E_r(\rho) = \alpha_{j-1} \ln(\frac{N}{N-2}\alpha_{j-1}) + \alpha_j \ln(\frac{N+1}{2}\alpha_j) + (1 - \alpha_{j-1} - \alpha_j) \ln[\frac{N(N+1)}{2}(1 - \alpha_{j-1} - \alpha_j)].$$

- If ρ is in the triangle $A'DH$,

$$E_r(\rho) = \alpha_{j-1} \ln \frac{(N+3)(N-1)\alpha_{j-1}}{(N^2-5)b} + \alpha_j \ln \frac{(N^2-1)(N^2-9)\alpha_j}{4N(N^2-5)b - 2(N+3)(N-1)^2}$$

$$+(1 - \alpha_{j-1} - \alpha_j) \ln \frac{(N+1)(N-3)(1-\alpha_{j-1}-\alpha_j)}{(N^2-5)(1-b)},$$

here

$$b = \frac{t_2 + \sqrt{t^2 - 8N(N^2-5)(N-1)^2(N+3)\alpha_{j-1}}}{4N(N^2-5)},$$

$$t_2 = (N+3)(N-1)^2 + 2N(N^2-5)\alpha_{j-1} + (N+1)^2(N-3)\alpha_j.$$

C. The case $3 \otimes N$ with even N

Since the set of the separable states is bounded by the straight lines AE, $A'E$ and a concave curve for a bipartite system of spin-1 particle and spin-j particle with half-integer j [11], it is cumbersome to compute the relative entropy of entanglement for this kind of RI states. However, we have some interesting results on RI states of $3 \otimes N$ system with even N. Considering $E_\Gamma(\rho)$, the state space for $3 \otimes N$ with even N RI states is also split into four regions: the PPT polygon $ADA'E$, the entangled triangle $A'DH$, the entangled polygon $A'HBF$ and the entangled polygon $A'FCE$.

Analogous to theorem 4 in [18], one can reduce to the property of $E_\Gamma(\rho)$: if σ^* minimizes $S(\rho\|\sigma^*)$ over $\sigma \in \mathcal{P}$ then σ^* is also a minimum for any state of the form $\rho_x = (1-x)\rho + x\sigma^*$. Thus making use of this property of $E_\Gamma(\rho)$ and the similar derivation to relative entropy of entanglement for RI state of $3 \otimes 5$ system, we obtain $E_\Gamma(\rho)$ for all rotationally invariant states of a spin-1 particle and a spin-j particle with half-integer j. $E_\Gamma(\rho')$ for RI state ρ' of $3 \otimes N$ system with even N has the same expression as $E_r(\rho)$ for RI state ρ of $3 \otimes N$ system with odd N in subsection IV B. According to the property (8) of relative entropy, $E_\Gamma(\rho)$ provides us a lower bound of relative entropy of entanglement and an upper bound on the rate at which entanglement can be distilled.

V. CONCLUSIONS

It was argued that the REE is the most appropriate quantity to measure distinguishability between different quantum states. Hence it could be a powerful tool for investigating quantum channels properties[20]. In the present paper, we give the formula of the REE for RI states of $2 \otimes M$ system and $3 \otimes N$ system with odd N. RI states of spin-1 particle and spin-j particle constitute a two-parameter family. Therefore, relative entropy of entanglement for them is a function of two variables. One can find that the expression of $E_r(\rho)$ for RI states of $3 \otimes N$ system with odd N can not be applied to the RI states of two spin-1

particles. It shows that RI states of two equal spins characterize distinct entanglement from RI states of two different spins. Although EoF for RI states is difficult to compute, we give a lower bound of EoF for RI states of spin-1 and arbitrary spin-j particles. Meanwhile, it is an upper bound for the number of singlet states that can be distilled from a given RI state. In addition, H.P. Breuer [11] points out that S^α_{PPT} approaches the set S^α meanwhile $S^\alpha_{sep}(= S^\alpha \cap \mathcal{D})$ approaches S^α_{PPT} as N increases. Thus we find that the relative entropy of entanglement vanishes for $3 \otimes N$ RI states when $N \to \infty$. Interestingly, the asymptotic relative entropy of entanglement with respect to positive partial transpose (AREEP) which is defined as the regularisation

$$E_r^\infty(\rho) = \lim_{n\to\infty} \frac{1}{n} E_r(\rho^{\otimes n}),$$

is investigated on Werner states and orthogonally invariant state as a sharper bound to distillable entanglement in [21, 22]. Therefore, we will investigate the AREEP on RI states in the future work.

VI.   ACKNOWLEDGEMENT

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